

# The gauge invariance of general relativistic tidal heating

Patricia Purdue

*Theoretical Astrophysics, California Institute of Technology, Pasadena, CA 91125*

(Sumbitted 29 January 1999; revised 15 June 1999)

When a self-gravitating body (e.g., a neutron star or black hole) interacts with an external tidal field (e.g., that of a binary companion), the interaction can do work on the body, changing its mass-energy. The details of this “tidal heating” are analyzed using the Landau-Lifshitz pseudotensor and the local asymptotic rest frame of the body. It is shown that the work done on the body is gauge-invariant, while the body-tidal-field interaction energy contained within the body’s local asymptotic rest frame is gauge dependent. This is analogous to Newtonian theory, where the interaction energy is shown to depend on how one localizes gravitational energy, but the work done on the body is independent of that localization.

PACS numbers: 04.20.cv, 04.25.-g, 04.40.Dg, 04.70.-s

## I. INTRODUCTION AND SUMMARY

This is one in a series of papers that develops perturbative mathematical and physical tools for studying the interaction of an isolated gravitating body with a complicated “external universe” in the slow-motion limit. By “slow-motion limit” we mean that the shortest timescale  $\tau$  for changes of the body’s multipole moments and/or changes of the universe’s tidal gravitational field is long compared to the body’s size  $R$ :  $R/\tau \ll 1$ , where we have set the speed of light equal to unity.

In addition to this slow-motion requirement, we also require that the body be isolated from other objects in the external universe, in the sense that both the radius of curvature  $\mathcal{R}$  of the external universe in the body’s vicinity, and the lengthscale  $\mathcal{L}$  on which the universe’s curvature changes there, are long compared to the body’s size:  $R/\mathcal{R} \ll 1$  and  $R/\mathcal{L} \ll 1$ .

The slow-motion, isolated-body formalism, to which this paper is a technical addendum, is, in essence, a perturbative expansion in powers of the small parameters  $R/\tau$ ,  $R/\mathcal{R}$  and  $R/\mathcal{L}$ . For a detailed discussion of the structure of this expansion and its realm of validity, see Thorne and Hartle [1]. As they discuss at length (their Sec. I.B), the slow-motion and isolated-body assumptions make no reference to the internal gravity of the object under study. Consequently, the Thorne-Hartle formalism in general, and the results of this paper in particular, can be applied even to strongly-gravitating bodies, as long as the source of the external tidal field is far enough away to allow a “buffer zone” where gravity is weak. This buffer zone, called the local asymptotic rest frame, will be described more fully at the beginning of Sec. III.

Two examples of isolated, slow-motion bodies are: (i) a neutron star or black hole in a compact binary system that spirals inward due to emission of gravitational waves; and (ii) the satellite Io, which travels around Jupiter in an elliptic orbit and gets heated by Jupiter’s tidal gravitational field [2].

The series of papers that has been developing the perturbative formalism for studying tidal effects in such slow-motion, isolated bodies is:

1. The book *Gravitation* [3], Section 20.6 (written by John Wheeler): laid the conceptual foundations for analyzing the motion of such an isolated body through the external universe.
2. Thorne and Hartle [1]: formulated the problem of analyzing the effects of the external universe’s tidal fields on such an isolated body, and conceived and initiated the development of the perturbative formalism for studying the influence of the tidal fields on the body’s motion through the external universe, the precession of its spin axis, and its changes of mass-energy.
3. Thorne [4]: developed the theory of multipole moments of the isolated body in the form used by Thorne and Hartle.
4. Zhang [5]: developed the theory of multipole mo-

ments for the external universe’s tidal gravitational fields, which underlies the work of Thorne and Hartle.

5. Zhang [6]: extended the Thorne-Hartle analysis of motion and precession to include higher order moments than they considered.

6. Thorne [7] and Flanagan [8]: initiated the study of tidally-induced volume changes in the isolated body, using the above formalism. Their studies were motivated by numerical solutions of Einstein’s equations by Wilson, Mathews, and Maronetti [9] which seemed to show each neutron star in a binary being compressed to the point of collapse by gravitational interaction with its companion. Thorne and Flanagan found no such effect of the large magnitude seen in the numerical solutions. An important piece of Thorne’s analysis came from examining the work done on each star by its companion’s tidal field—i.e., an analysis of “tidal heating.”

Thorne’s analysis of tidal heating required dealing with an issue that Thorne and Hartle had discussed, but avoided confronting: For an isolated body with mass quadrupole moment  $\mathcal{I}_{jk}$ , being squeezed by an external tidal gravitational field  $\mathcal{E}_{jk} \equiv R_{j0k0}$  (with  $R_{\alpha\beta\gamma\delta}$  the external Riemann tensor), there appears to be an ambiguity in the body’s total mass-energy  $M$  of order  $\delta M \sim \mathcal{I}_{jk}\mathcal{E}_{jk}$ . (Here and throughout we use locally Cartesian coordinates in the body’s local asymptotic rest frame; cf. Thorne and Hartle [1]. Because the coordinates are Cartesian, it makes no difference whether tensor indices are placed up or down.)

One can understand this apparent ambiguity by examining the time-time part of the spacetime metric in the body’s local asymptotic rest frame:

$$g_{00} = -1 + \frac{2M}{r} + \frac{3\mathcal{I}_{jk}n^jn^k}{r^3} + \dots - \mathcal{E}_{jk}n^jn^kr^2 + \dots \quad (1)$$

Here  $n^j = x^j/r$  is the unit radial vector and  $r$  is distance from the body in its local asymptotic rest frame. Among the terms omitted here are those of quadratic and higher order in the body’s mass  $M$  and quadrupole moment  $\mathcal{I}_{jk}$  and the external tidal field  $\mathcal{E}_{jk}$ —terms induced by nonlinearities of the Einstein field equations. Among these nonlinear terms is

$$\delta g_{00} \sim \frac{\mathcal{I}_{jk}\mathcal{E}_{jk}}{r}, \quad (2)$$

whose  $1/r$  behavior can be deduced by dimensional considerations. This term has the multipolar structure (monopole)/ $r$  identical to that of the  $2M/r$  term from which one normally reads off the body’s total mass, and its numerical coefficient is ambiguous corresponding to the possibility to move some arbitrary portion of it into or out of the  $2M/r$  term. Correspondingly, the body’s mass is ambiguous by an amount of order

$$\delta M \sim \mathcal{I}_{jk}\mathcal{E}_{jk}. \quad (3)$$

In Sec. II of this paper we shall see that this ambiguity is not a purely relativistic phenomenon. In the Newtonian theory of gravity, there is also an ambiguity of magnitude (3) in that portion of the gravitational interaction energy of the body and external field which is contained within the body's local asymptotic rest frame. In Sec. III we shall explore the relativistic version of this “tidal-quadrupolar” interaction ambiguity by computing how the interaction energy changes under a relativistic change of gauge (infinitesimal coordinate transformation).

“Tidal heating” of the isolated body involves injecting into it an amount of energy that is of just the same magnitude as this ambiguity,  $\Delta M \sim \mathcal{I}_{jk} \mathcal{E}_{jk}$ . Does this mean that tidal heating is an ill-defined, unphysical concept? Certainly not, as is attested by photographs of volcanic plumes ejected from Jupiter's satellite Io (see, e.g., Ref. [10]). That volcanism was predicted by Peale, Cassen, and Reynolds [2] before the Voyager spacecraft discovered it; and their explanation—that Io gets tidally heated to high internal temperature by the coupling of its quadrupole moment to Jupiter's tidal field—is firm (see, e.g., Refs. [11,12]).

In this paper, we use the phrase “tidal heating” to mean the net work done by an external tidal field on an isolated body. This phrase is slightly misleading, as the work done on the body need not necessarily go only into heat. The additional energy might be used to deform the body (i.e., raise a tide on it) or it might go into vibrational energy. If, however, timescales for changes of  $\mathcal{E}_{jk}$  and  $\mathcal{I}_{jk}$  are not close to the body's vibrational periods, then, when averaged over many cycles of change of  $\mathcal{E}_{jk}$  and/or  $\mathcal{I}_{jk}$ , the work will contribute primarily to heat, as in the case of Io.

In his analysis of binary neutron star systems, Thorne [7] argued, but did not prove in detail, that although the tidal-quadrupole interaction energy is ambiguous, the amount of work done on an isolated body by an external tidal field (i.e., the amount of tidal heating) is unambiguously given by the formula<sup>1</sup>

$$\frac{dW}{dt} = -\frac{1}{2} \mathcal{E}_{ij} \frac{d\mathcal{I}_{ij}}{dt}; \quad (4)$$

and he argued that this is true in general relativity theory as well as in Newtonian theory. In Sec. II we shall give a Newtonian demonstration of this claim. In Sec. III we shall give a relativistic demonstration—showing, more specifically, that although the quadrupole/tidal interaction energy is gauge dependent, the work done is gauge invariant and has the value (4).

---

<sup>1</sup> Actually, expression (4) is only the leading order term in the perturbative expansion of  $dW/dt$ . The next term is  $-\frac{2}{3} \mathcal{B}_{ij} d\mathcal{S}_{ij}/dt$ , where  $\mathcal{B}_{ij}$  is the “magnetic type” tidal field of the external universe and  $\mathcal{S}_{ij}$  is the body's current quadrupole moment [6]. In this paper we confine attention to the leading-order term.

## II. NEWTONIAN ANALYSIS

In this section, we consider a Newtonian body, with weak internal gravity  $|\Phi_o| \ll 1$  (where  $\Phi_o$  is the body's gravitational potential), subjected to an external Newtonian tidal field. We assume that the external field is nearly homogeneous in the vicinity of the body,  $\mathcal{L} \gg R$  (cf. Fig. 1; in the Newtonian case, the inner boundary of the vacuum “local asymptotic rest frame”<sup>2</sup>—indicated by the inner dashed circle—would be at the surface of the body).

In our analysis, we will consider a variety of contributions to the total energy inside a sphere which encompasses the body and whose boundary lies within the local asymptotic rest frame—i.e., the region where the external field is nearly homogeneous (again, cf. Fig. 1). We denote by  $\mathcal{V}$  the interior of this sphere and by  $\partial\mathcal{V}$  its boundary. Of greatest interest will be the interaction energy (between the body and the external tidal field) and the work done by the tidal field on the body. Both of these quantities are the result of slow changes of the tidal field  $\mathcal{E}_{jk}$  and the body's quadrupole moment  $\mathcal{I}_{jk}$ .

As a foundation for our analysis, consider a fully isolated system that includes the body of interest and other “companion” bodies, which produce the tidal field  $\mathcal{E}_{jk}$  that the body experiences. For simplicity, assume that all the bodies are made of perfect fluid (a restriction that can easily be abandoned). Then, for the full system, the Newtonian gravitational energy density and energy flux can be written as

$$\Theta_1^{00} = \rho \left( \frac{1}{2} v^2 + \Pi + \Phi \right) + \frac{1}{8\pi} \Phi_{,j} \Phi_{,j}, \quad (5)$$

$$\Theta_1^{0j} = \rho v^j \left( \frac{1}{2} v^2 + \Pi + \frac{p}{\rho} + \Phi \right) - \frac{1}{4\pi} \Phi_{,t} \Phi_{,j}, \quad (6)$$

where  $\Phi, \rho, p, v$ , and  $\Pi$  are the Newtonian gravitational potential, mass density, pressure, velocity, and specific internal energy [7].

Using conservation of rest mass  $\rho_{,t} + (\rho v^j)_{,j} = 0$ , the first law of thermodynamics  $\rho d\Pi/dt + p v^j_{,j} = 0$ , the Euler equation for fluids  $\rho dv^j/dt + \rho \Phi_{,j} + p_{,j} = 0$ , Newton's field equation  $\Phi_{,jj} = 4\pi\rho$ , and the definition of the comoving time derivative  $d/dt = \partial/\partial t + v^j \partial/\partial x^j$ , it can be shown that Eqs. (5) and (6) satisfy conservation of energy

$$\Theta^{00}_{,t} + \Theta^{0j}_{,j} = 0. \quad (7)$$

Equations (5) and (6) for the energy density and flux, however, are not unique. Equally valid are the following expressions:

---

<sup>2</sup>We shall discuss the concept of “local asymptotic rest frame” near the beginning of Sec. III.

$$\Theta_2^{00} = \rho \left( \frac{1}{2} v^2 + \Pi \right) - \frac{1}{8\pi} \Phi_{,j} \Phi_{,j} , \quad (8)$$

$$\Theta_2^{0j} = \rho v^j \left( \frac{1}{2} v^2 + \Pi + \frac{p}{\rho} + \Phi \right) + \frac{1}{4\pi} \Phi_{,tj} \Phi , \quad (9)$$

which also satisfy energy conservation (7) but localize the gravitational energy in a different manner from  $\Theta_1^{0\mu}$ . Energy conservation (7) will also be satisfied by any linear combination of  $\Theta_1^{0\mu}$  and  $\Theta_2^{0\mu}$ . Imposing the additional condition that, for any acceptable  $\Theta$ , the system's total energy

$$E = \int \Theta^{00} d^3x \quad (10)$$

must be independent of the choice of  $\Theta^{00}$  forces the coefficients to sum to 1. Hence, a perfectly valid form for  $\Theta^{0\mu}$  is

$$\Theta^{0\mu} = \alpha \Theta_1^{0\mu} + (1 - \alpha) \Theta_2^{0\mu} , \quad (11)$$

where  $\alpha$  is an arbitrary constant.

Notice that the choice of  $\alpha$  gives a specific energy localization. For example,  $\alpha = 0$  puts the gravitational energy entirely in the field  $[-(\nabla\Phi)^2/(8\pi)]$ , so it is non-zero outside the matter. This is analogous to the localization used in electrostatics ( $1/8\pi$  times the square of the electric field). Choosing  $\alpha = \frac{1}{2}$ , by contrast, puts the gravitational energy entirely in the matter ( $\frac{1}{2}\rho\Phi$ ), so it vanishes outside the body. When  $\frac{1}{2}\rho\Phi$  is integrated over the entire system (body plus its companions), the result is a widely used way of computing gravitational energy (e.g., Sec. 17.1 of Ref. [13]).

The energy in the region  $\mathcal{V}$  that contains and surrounds the body but excludes the companion,

$$E_{\mathcal{V}} = \int_{\mathcal{V}} \Theta^{00} d^3x , \quad (12)$$

will depend on  $\alpha$ ; i.e., it will depend on where the energy is localized. By contrast, the total energy (10) for the fully isolated system (body plus its companions) will be independent of  $\alpha$ .

Another way to express this ambiguity of the localization of the gravitational energy is given by Thorne (Appendix of Ref. [7]): it is possible to add the divergence of  $\eta_j = \beta\Phi\Phi_{,j}$  (where  $\beta$  is an arbitrary constant) to  $\Theta_1^{00}$  and the time derivative of  $-\eta_j$  to  $\Theta_1^{0j}$  without affecting energy conservation (7) or the physics of the system. Indeed, this method is completely equivalent to the one presented above. The constants are related by  $\beta = (\alpha - 1)/4\pi$ .

Throughout the region  $\mathcal{V}$ , the Newtonian gravitational potential can be broken into two parts: the body's self field  $\Phi_o$  and the tidal field  $\Phi_e$  produced by the external universe (the companion bodies), so that

$$\Phi = \Phi_o + \Phi_e . \quad (13)$$

The external field is quadrupolar and source-free in the region  $\mathcal{V}$  so that

$$\Phi_e = \frac{1}{2} \mathcal{E}_{ij} x^i x^j , \quad \Phi_{e,jj} = 0 , \quad (14)$$

and, furthermore, the tidal field  $\mathcal{E}_{ij}$  evolves slowly with time. The body's (external) self field is monopolar and quadrupolar and has the body's mass distribution as a source:

$$\Phi_o = -\frac{M}{r} - \frac{3}{2} \frac{\mathcal{I}_{ij} n^i n^j}{r^3} , \quad \Phi_{o,jj} = 4\pi\rho . \quad (15)$$

The quadrupole moment  $\mathcal{I}_{ij}$ , like that of the external field, evolves slowly with time, but the body's mass  $M$  is constant. Recall that  $r \equiv \sqrt{\delta_{ij} x^i x^j}$  is radial distance from the body's center of mass and  $n^j \equiv x^j/r$  is the radial vector.

A useful expression for the total energy  $E_{\mathcal{V}}$  in the spherical region  $\mathcal{V}$  can be derived by inserting Eqs. (11), (5), (8), and (13) into Eq. (12). The resulting expression can be broken into a sum of three parts—the body's self energy  $E_o$  (which depends only on  $\Phi_o$  and  $\rho$ ), the external field energy  $E_e$  (which depends only on  $\Phi_e$ ), and the interaction energy  $E_{\text{int}}$  (which involves products of  $\Phi_e$  with  $\rho$  or  $\Phi_o$ ):

$$E_{\mathcal{V}} = E_o + E_e + E_{\text{int}} , \quad (16)$$

where

$$E_o = \int_{\mathcal{V}} \left[ \rho \left( \frac{1}{2} v^2 + \Pi \right) + \alpha \rho \Phi_o + \frac{(2\alpha - 1)}{8\pi} \Phi_{o,j} \Phi_{o,j} \right] d^3x , \quad (17)$$

$$E_e = \int_{\mathcal{V}} \left[ \frac{(2\alpha - 1)}{8\pi} \Phi_{e,j} \Phi_{e,j} \right] d^3x , \quad (18)$$

$$E_{\text{int}} = \int_{\mathcal{V}} \left[ \alpha \rho \Phi_e + \frac{(2\alpha - 1)}{4\pi} \Phi_{o,j} \Phi_{e,j} \right] d^3x . \quad (19)$$

Inserting Eqs. (14) and (15) into Eq. (19) and integrating gives the interaction energy inside  $\mathcal{V}$  as

$$E_{\text{int}} = \frac{(2 + \alpha)}{10} \mathcal{E}_{ij} \mathcal{I}_{ij} , \quad (20)$$

which depends on  $\alpha$ . In other words, it depends on our arbitrary choice of how to localize gravitational energy. This is the ambiguity of the interaction energy discussed in Sec. I.

The rate of change of the total energy inside  $\mathcal{V}$  can be expressed in the form

$$\frac{dE_{\mathcal{V}}}{dt} = - \int_{\partial\mathcal{V}} \Theta^{0j} n^j r^2 d\Omega \quad (21)$$

by taking the time derivative of Eq. (12), inserting Eq. (7), and applying the divergence theorem. This expression, like that for the energy, can be broken into a sum by combining Eqs. (11), (6), (9), (13), (14), (15), and (20):

$$\frac{dE_{\mathcal{V}}}{dt} = \frac{dE_e}{dt} + \frac{dE_{\text{int}}}{dt} - \frac{1}{2}\mathcal{E}_{ij}\frac{d\mathcal{I}_{ij}}{dt} + \int_{\partial\mathcal{V}} \frac{1}{4\pi} [\alpha\Phi_{o,t}\Phi_{o,j} + (\alpha-1)\Phi_{o,tj}\Phi_o] n^j r^2 d\Omega. \quad (22)$$

The first term is the rate of change of the external field energy (18) inside  $\mathcal{V}$ , resulting from the evolution of the tidal field. The second term is the rate of change of the interaction energy (20). The third and fourth terms together, by comparison with Eq. (16), must be equal to  $dE_o/dt$ , the rate of change of the body's self energy. The fourth term is the contribution from the body's own field moving across the boundary  $\partial\mathcal{V}$ . Therefore, the third term gives the change in the body's energy coming from the interaction with the tidal field; in other words, it is the rate of work done on the body by the tidal field. Furthermore, this term is independent of  $\alpha$  and is, consequently, independent of how the Newtonian energy is localized, as claimed in Sec. I, Eq. (4).

### III. RELATIVISTIC ANALYSIS

In this section, we will exhibit a relativistic version of the calculation in Sec. II, again showing that the rate of work done on the body by the tidal field is gauge invariant and that it has the same value in a general relativistic perturbative treatment as in the Newtonian one:  $-(1/2)\mathcal{E}_{ij}d\mathcal{I}_{ij}/dt$ . The formalism to be used is the Landau-Lifshitz energy-momentum pseudotensor and multipole expansions as developed by Thorne and Hartle [1] and Zhang [5,6], together with the slow-motion approximation, so time derivatives are small compared to spatial gradients.

We will work in the body's vacuum local asymptotic rest frame, which is defined as a region outside the body and far enough from it that its gravitational field can be regarded as a weak perturbation of flat spacetime, yet near enough that the tidal field of the external universe can be regarded as nearly homogeneous. This region is a spherical shell around the body; its inner boundary is near the body's surface but far enough away for gravity to be weak, and its outer boundary is at a distance where the external tidal field begins to depart from homogeneity (see Fig. 1). Somewhat more precisely, the local asymptotic rest frame is the region throughout which  $r/\mathcal{L} \ll 1$ ,  $r/\mathcal{R} \ll 1$ , and  $M/r \ll 1$ , where  $r$  is the radial distance from the body,  $M$  is the mass of the body, and  $\mathcal{R}$  and  $\mathcal{L}$  are the radius of curvature and the scale of homogeneity of the external gravitational field. If this region exists (as in the case of a black hole binary far from merger, for example) and the slow-motion limit applies, then the following analysis is valid.

As in the Newtonian case, we will consider contributions to the total energy inside a sphere  $\mathcal{V}$  which encompasses the body and whose boundary  $\partial\mathcal{V}$  lies within the local asymptotic rest frame (see Fig. 1).

#### A. deDonder Gauge

We shall begin by computing the work done on the body by the external tidal field, using deDonder gauge (this section). Then in the next section, we shall show that the work is gauge invariant.

DeDonder gauge is defined by the condition that  $\bar{h}^{\alpha\beta}_{;\beta} = 0$ , where  $\bar{h}^{\alpha\beta}$  is defined in terms of the metric density as follows:

$$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} \equiv \eta^{\alpha\beta} - \bar{h}^{\alpha\beta}. \quad (23)$$

At linear order in the strength of gravity,  $\bar{h}^{\alpha\beta}$  is the trace-reversed metric perturbation. According to Zhang [6], the components of  $\bar{h}^{\alpha\beta}$  in the body's local asymptotic rest frame are, at leading (linear) order in the strength of gravity and at leading non-zero order in our slow-motion expansion,

$$\bar{h}^{00} \equiv -4\Phi = 4\frac{M}{r} + 6\frac{\mathcal{I}_{ij}x^i x^j}{r^5} - 2\mathcal{E}_{ij}x^i x^j, \quad (24)$$

$$\bar{h}^{0j} \equiv -A_j = 6\frac{\dot{\mathcal{I}}_{ja}x^a}{r^3} + \frac{10}{21}\dot{\mathcal{E}}_{ab}x^a x^b x^j - \frac{4}{21}\dot{\mathcal{E}}_{ja}x^a r^2, \quad (25)$$

$$\bar{h}^{ij} = \mathcal{O}\left(\ddot{\mathcal{E}}_{ij}r^4 \text{ \& } \frac{\ddot{\mathcal{I}}_{ij}}{r}\right), \quad (26)$$

where the dots indicate time derivatives (i.e.,  $\dot{\mathcal{I}}_{ij} \equiv d\mathcal{I}_{ij}/dt$ ) and the symbol “&” means “plus terms of the form and magnitude.” Note that the higher-order ( $\ell$ -order) multipoles have been omitted, since their contributions are smaller by  $\sim (r/\mathcal{L})^{\ell-2} \leq r/\mathcal{L} \ll 1$  and  $\sim (M/r)^{\ell-2} \leq M/r \ll 1$  than the quadrupolar ( $\ell = 2$ ) terms that we have kept. The second time derivatives will also be neglected since they are unimportant in the slow-motion approximation; this effectively eliminates  $\bar{h}^{ij}$  in this gauge. Also, note that the  $\Phi$  in Eq. (24) has the same form as the Newtonian gravitational potential of Sec. II, and the  $A_j$  of Eq. (25) is a gravitational vector potential, which does not appear in Newtonian theory.

In general, it would be possible to have a term  $\propto \mathcal{I}_{jk}\mathcal{E}_{jk}/r$  in Eq. (24), as well as the  $\propto M/r$  term. We have chosen to define the constant for the monopolar term to be  $4M$ , thereby eliminating any term  $\propto \mathcal{I}_{jk}\mathcal{E}_{jk}/r$ ; this is arbitrary but convenient, as will be seen shortly.

The total mass-energy  $\mathcal{M}_{\mathcal{V}}$  inside the sphere  $\mathcal{V}$  (total stellar mass including the quadrupolar deformation energy and energy of interaction between the deformation and tidal field) is defined by Thorne and Hartle (Eq. (2.2a) of Ref. [1]) in terms of the Landau-Lifshitz superpotential as

$$\mathcal{M}_{\mathcal{V}} = \frac{1}{16\pi} \int_{\partial\mathcal{V}} H^{0\alpha 0j}_{;\alpha} n_j r^2 d\Omega, \quad (27)$$

where

$$H^{\mu\alpha\nu\beta} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}; \quad (28)$$

cf. Eqs. (20.6), (20.3), and (20.20) of MTW [3] and Eqs. (100.14) and (100.2) of Landau and Lifshitz [14]. Using Eqs. (28), (23), and (24)–(26) in Eq. (27) and carrying out the integration gives

$$\begin{aligned}\mathcal{M}_V &= M + \mathcal{O}\left(\mathcal{E}\ddot{\mathcal{E}}r^7 \text{ \& } \dot{\mathcal{E}}\dot{\mathcal{E}}r^7\right) \\ &+ \mathcal{O}\left(\dot{\mathcal{I}}\dot{\mathcal{E}}r^2 \text{ \& } \ddot{\mathcal{I}}\mathcal{E}r^2 \text{ \& } \mathcal{I}\ddot{\mathcal{E}}r^2\right) \\ &+ \mathcal{O}\left(\frac{\mathcal{I}\ddot{\mathcal{I}}}{r^3} \text{ \& } \frac{\dot{\mathcal{I}}\dot{\mathcal{I}}}{r^3}\right). \quad (29)\end{aligned}$$

Hence,  $\mathcal{M}_V = M$  at leading order in our slow-motion approximation when we neglect the double and higher-order time derivatives. This is the reason the constant of the monopolar term in Eq. (24) was chosen to be  $4M$ .

To calculate the rate of work done by the tidal field on the body when  $\mathcal{E}_{jk}$  and  $\mathcal{I}_{jk}$  change slowly, we consider the change of the mass-energy  $\mathcal{M}_V$ ,

$$-\frac{d\mathcal{M}_V}{dt} = \int_{\partial V} (-g)t^{0j}n_j r^2 d\Omega, \quad (30)$$

where  $t^{\mu\nu}$  is the Landau-Lifshitz pseudotensor. That this expression is indeed the time derivative of Eq. (27) is a result of Gauss's theorem (see discussion after Eqs. (2.3) of Ref. [1]).

In the body's local asymptotic rest frame, the Landau-Lifshitz energy-momentum pseudotensor (Eq. (20.22) in MTW [3]) is, at the orders of accuracy we are considering,

$$\begin{aligned}(-g)t^{00} &= \frac{1}{16\pi} \left(-\frac{7}{8}\right) g^{ij}\bar{h}^{00}_{,i}\bar{h}^{00}_{,j} \\ &= -\frac{7}{8\pi} \delta^{ij} \Phi_{,i} \Phi_{,j}, \quad (31)\end{aligned}$$

$$\begin{aligned}(-g)t^{0j} &= \frac{1}{16\pi} \left(\frac{3}{4}\bar{h}^{00}_{,0}\bar{h}^{00}_{,j} + \bar{h}^{00}_{,m}\bar{h}^{0m}_{,j} - \bar{h}^{00}_{,m}\bar{h}^{0j}_{,m}\right) \\ &= \frac{3}{4\pi} \Phi_{,0}\Phi_{,j} + \frac{1}{4\pi} (A_{k,j} - A_{j,k}) \Phi_{,k}. \quad (32)\end{aligned}$$

Here the deDonder gauge condition  $\bar{h}^{\alpha\beta}_{,\beta} = 0$  has been used to eliminate many terms from the general expression in MTW, but most of the simplification has come from keeping only terms of leading-order in the slow-motion approximation [zeroth and first time derivatives, respectively, in Eqs. (31) and (32)]. This restriction has given us only products of  $\bar{h}^{\mu\nu}_{,\alpha}$  which will produce terms containing the products  $M^2$ ,  $M\mathcal{E}$ ,  $M\mathcal{I}$ ,  $\mathcal{I}\mathcal{E}$ ,  $\mathcal{I}\mathcal{I}$ ,  $\mathcal{E}\mathcal{E}$  for  $(-g)t^{00}$  and  $M\dot{\mathcal{I}}$ ,  $M\dot{\mathcal{E}}$ ,  $\mathcal{I}\dot{\mathcal{E}}$ ,  $\mathcal{E}\dot{\mathcal{I}}$ ,  $\mathcal{I}\dot{\mathcal{I}}$ ,  $\mathcal{E}\dot{\mathcal{E}}$  for  $(-g)t^{0j}$ . This may be illustrated by expressing  $(-g)t^{0\alpha}$  explicitly in terms of the quadrupole moments by substituting Eqs. (24) and (25) into Eqs. (31) and (32) to get

$$\begin{aligned}(-g)t^{00} &= \frac{1}{16\pi} \left(-14\frac{M^2}{r^4} - 210\frac{\mathcal{I}_{ab}\mathcal{E}_{cd}x^a x^b x^c x^d}{r^7}\right. \\ &\quad \left.+ 84\frac{\mathcal{I}_{ab}\mathcal{E}_{bc}x^a x^c}{r^5} - 28\frac{M\mathcal{E}_{ab}x^a x^b}{r^3} - 14\mathcal{E}_{ab}\mathcal{E}_{bc}x^a x^c\right.\end{aligned}$$

$$\begin{aligned}&\left.- 126\frac{M\mathcal{I}_{ab}x^a x^b}{r^8} - 126\frac{\mathcal{I}_{ab}\mathcal{I}_{bc}x^a x^c}{r^{10}}\right. \\ &\quad \left.- \frac{315}{2}\frac{\mathcal{I}_{ab}\mathcal{I}_{cd}x^a x^b x^c x^d}{r^{12}}\right), \quad (33)\end{aligned}$$

and

$$\begin{aligned}(-g)t^{0j} &= \frac{1}{16\pi} \left(-18\frac{\dot{\mathcal{I}}_{ab}\mathcal{E}_{cj}x^a x^b x^c}{r^5} + 24\frac{\dot{\mathcal{I}}_{ab}\mathcal{E}_{bc}x^a x^c x^j}{r^5}\right. \\ &\quad - 24\frac{\dot{\mathcal{I}}_{aj}\mathcal{E}_{bc}x^a x^b x^c}{r^5} - 18\frac{\mathcal{I}_{aj}\dot{\mathcal{E}}_{bc}x^a x^b x^c}{r^5} \\ &\quad - 24\frac{\mathcal{I}_{ab}\dot{\mathcal{E}}_{cj}x^a x^b x^c}{r^5} - 16\frac{\mathcal{I}_{ab}\dot{\mathcal{E}}_{bc}x^a x^c x^j}{r^5} \\ &\quad \left.+ 85\frac{\mathcal{I}_{ab}\dot{\mathcal{E}}_{cd}x^a x^b x^c x^d x^j}{r^7}\right). \quad (34)\end{aligned}$$

Note that we have kept only the  $\mathcal{E}\mathcal{I}$  cross terms in the expression for  $(-g)t^{0j}$ , as only they will contribute to our calculation of the interaction energy and work.

We find the rate of change of mass-energy inside our sphere  $V$  by inserting Eq. (34) into Eq. (30) and integrating. The result (only considering the cross terms) is

$$-\frac{d\mathcal{M}_V}{dt} = \frac{d}{dt} \left(\frac{1}{10}\mathcal{E}_{ij}\mathcal{I}_{ij}\right) + \frac{1}{2}\mathcal{E}_{ij}\frac{d}{dt}\mathcal{I}_{ij}. \quad (35)$$

Since the interaction energy can depend only on the instantaneous deformation and tidal field, its rate of change must be a perfect differential, whereas the rate of work need not be. Also, if the tidal field changes but the body does not, there is no work done. From these two facts, we can conclude that the first term of Eq. (35) is the rate of change of the interaction energy between the tidal field and the body and that the second term represents the rate of work done by the external field on the body (the “tidal heating”).

Notice that this value for the rate of work matches that discussed in Sec. I, Eq. (4), and found via the Newtonian analysis in Sec. II. Note the comparison with our Newtonian expressions (22) and (20). The first and fourth terms of Eq. (22) are missing here because we included in our  $(-g)t^{0j}$  only the (body field)  $\times$  (external field) crossterms. If we had also included (external)  $\times$  (external) and (body)  $\times$  (body) terms, Eq. (35) would have entailed expressions like the first and fourth terms of Eq. (22). Note also from the interaction energy terms in Eqs. (22), (20), and (35) that the Landau-Lifshitz way of localizing gravitational energy corresponds to the Newtonian choice  $\alpha = -3$ , a correspondence that has previously been derived by Chandrasekhar [15].

## B. General Gauge

In Sec. III. A., we considered the special case of deDonder gauge, which was particularly simple due to the

gauge condition and to the  $\bar{h}^{ij}$  terms being effectively zero. Now we will examine a general gauge, which can be achieved by a gauge transformation of the form

$$\begin{aligned}\bar{h}_{\mu\nu} &\rightarrow \bar{h}_{\mu\nu} + \delta\bar{h}_{\mu\nu}; \\ \delta\bar{h}_{\mu\nu} &= \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\xi^\alpha_{,\alpha},\end{aligned}\quad (36)$$

where  $\xi^\alpha$  is a function to be chosen shortly. Using  $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\sigma_\sigma$ , where  $h_{\mu\nu}$  is the perturbation of the metric away from the Minkowski metric in the local asymptotic rest frame, this can also be expressed as

$$\begin{aligned}h_{\mu\nu} &\rightarrow h_{\mu\nu} + \delta h_{\mu\nu}; \\ \delta h_{\mu\nu} &= \xi_{\mu,\nu} + \xi_{\nu,\mu}.\end{aligned}\quad (37)$$

Note that in the deDonder gauge,

$$h_{00} = 2\frac{M}{r} + 3\frac{\mathcal{I}_{ij}x^ix^j}{r^5} - \mathcal{E}_{ij}x^ix^j, \quad (38)$$

$$h_{0j} = -6\frac{\dot{\mathcal{I}}_{ja}x^a}{r^3} - \frac{10}{21}\dot{\mathcal{E}}_{ab}x^ax^bx^j + \frac{4}{21}\dot{\mathcal{E}}_{ja}x^ar^2, \quad (39)$$

$$h_{ij} = \delta_{ij}\left(2\frac{M}{r} + 3\frac{\mathcal{I}_{k\ell}x^kx^\ell}{r^5} - \mathcal{E}_{k\ell}x^kx^\ell\right). \quad (40)$$

Since we are interested only in gauge changes of the same order as we have been using so far (leading-order in the slow-motion approximation), we include only terms in  $\xi_\alpha$  that will produce  $\delta h_{\mu\nu}$  of the same forms as Eqs. (38)–(40), but with different numerical coefficients. For example, consider  $\delta h_{00} = 2\xi_{0,0} \propto M/r$ ; that gives  $\xi_0 \propto Mt/r$ , since  $M$  is a constant. This, in turn, implies  $\delta h_{0j} = \xi_{0,j} \propto Mtx^j/r^3$ , but this is not of the same form as the terms in Eq. (39); rather, it corresponds to a gauge that rapidly becomes ill-behaved as time passes. Similar arguments apply to  $\delta h_{00} \propto \mathcal{I}_{jk}x^jx^k/r^5$  or  $\delta h_{00} \propto \mathcal{E}_{jk}x^jx^k$ ; their coefficients cannot be altered by a gauge change because such a change would alter the mathematical form of  $h_{0j}$  and would make its magnitude unacceptably large in the slow-motion limit. As a result, we must set  $\xi_0 = 0$ . If we now consider  $\delta h_{0j} = \xi_{j,0}$ , possible terms are of the form  $\propto \dot{\mathcal{I}}_{ja}x^a/r^3$ ,  $\propto \dot{\mathcal{E}}_{ab}x^ax^bx^j$ , or  $\propto \dot{\mathcal{E}}_{ja}x^ar^2$ ; cf. Eq. (39). Each of these gives  $\delta h_{00} = 0$  and  $\delta h_{ij}$  of the same form as Eq. (40); hence, terms of these forms are allowed. Consequently, the most general gauge change that preserves the mathematical forms of  $h_{\mu\nu}$  but alters their numerical coefficients is

$$\begin{aligned}\xi_0 &= 0, \\ \xi_j &= \alpha\frac{\mathcal{I}_{ja}x^a}{r^3} + \beta\mathcal{E}_{ja}x^ar^2 + \gamma\mathcal{E}_{ab}x^ax^bx^j,\end{aligned}\quad (41)$$

where  $\alpha, \beta$ , and  $\gamma$  are arbitrary constants.

Using the trace-reversed gauge change (36) with Eq. (41), the new  $\bar{h}^{\alpha\beta}$  become

$$\begin{aligned}\bar{h}^{00} &= 4\frac{M}{r} + (5\gamma + 2\beta - 2)\mathcal{E}_{ij}x^ix^j \\ &\quad + (6 - 3\alpha)\mathcal{I}_{ij}\frac{x^ix^j}{r^5},\end{aligned}\quad (42)$$

$$\begin{aligned}\bar{h}^{0j} &= (2 - \alpha)\dot{\mathcal{I}}_{ja}\frac{x^a}{r^3} + \left(\frac{10}{21} - \gamma\right)\dot{\mathcal{E}}_{ab}x^ax^bx^j \\ &\quad - \left(\frac{4}{21} + \beta\right)\dot{\mathcal{E}}_{ja}x^ar^2,\end{aligned}\quad (43)$$

$$\begin{aligned}\bar{h}^{jk} &= 2\alpha\mathcal{I}_{jk}\frac{1}{r^3} + 3\alpha\delta^{jk}\mathcal{I}_{ab}\frac{x^ax^b}{r^5} - 3\alpha\mathcal{I}_{ja}\frac{x^ax^k}{r^5} \\ &\quad - 3\alpha\mathcal{I}_{ka}\frac{x^ax^j}{r^5} + 2\beta\mathcal{E}_{jk}r^2 - 2(\beta + \gamma)\delta^{jk}\mathcal{E}_{ab}x^ax^b \\ &\quad + 2(\beta + \gamma)\mathcal{E}_{ja}x^ax^k + 2(\beta + \gamma)\mathcal{E}_{ka}x^ax^j.\end{aligned}\quad (44)$$

In this general gauge, if we calculate the total mass-energy  $\mathcal{M}_\mathcal{V}$  inside the sphere  $\mathcal{V}$  using Eqs. (28), (23), and (42)–(44) in Eq. (27), we find

$$\begin{aligned}\mathcal{M}_\mathcal{V} &= M + \left(2\gamma^2 + \frac{29}{5}\beta\gamma - \frac{4}{5}\gamma + 2\beta^2 - 2\beta\right)\mathcal{E}_{ij}\mathcal{E}_{ij}r^5 \\ &\quad + \left(\frac{1}{3}\alpha + 2\beta + \frac{7}{5}\gamma - \frac{4}{3}\alpha\beta - \frac{23}{15}\alpha\gamma\right)\mathcal{I}_{ij}\mathcal{E}_{ij}.\end{aligned}\quad (45)$$

The  $\mathcal{I}_{ij}\mathcal{I}_{ij}/r^5$  term is zero. It is comforting to note that all the new terms vanish for  $\alpha = \beta = \gamma = 0$ , giving the deDonder result  $\mathcal{M}_\mathcal{V} = M$ . The  $\mathcal{E}\mathcal{E}r^5$  and  $\mathcal{I}\mathcal{E}$  terms in Eq. (45) are large compared to the double time-derivative terms that formed the largest corrections to the mass-energy in deDonder gauge (29); however, they are still small compared to the mass  $M$  that appears in the expansion (24) of  $\bar{h}^{00}$ . Also note that, near the body of interest, the  $\mathcal{E}\mathcal{E}r^5$  term will be small compared to the  $\mathcal{I}\mathcal{E}$  term, due to its radial dependence. So, once again, we have  $\mathcal{M}_\mathcal{V} \simeq M$  as a first approximation, although it is necessary to keep the extra terms in Eq. (45) to maintain the same level of accuracy as we have been using. Consequently,  $\mathcal{M}_\mathcal{V}$  is gauge-dependent to the order that interests us, and it has the “ $\mathcal{I}_{ij}\mathcal{E}_{ij}$ ” ambiguity discussed by Thorne and Hartle [1] and mentioned in Sec. I.

Keeping only the leading-order terms in the slow-motion approximation, as described in Sec. III. A., the Landau-Lifshitz pseudotensor in the new gauge is

$$\begin{aligned}(-g)t^{00} &= \frac{1}{16\pi}\left(-\frac{7}{8}g^{ij}\bar{h}^{00}_{,i}\bar{h}^{00}_{,j} + \bar{h}^{00}_{,i}\bar{h}^{ij}_{,j} \right. \\ &\quad \left. - \frac{1}{2}\bar{h}^{ij}_{,k}\bar{h}^{ik}_{,j} + \frac{1}{4}\bar{h}^{00}_{,i}\bar{h}^{jj}_{,i} \right. \\ &\quad \left. + \frac{1}{4}\bar{h}^{ij}_{,k}\bar{h}^{ij}_{,k} - \frac{1}{8}\bar{h}^{ii}_{,k}\bar{h}^{jj}_{,k}\right),\end{aligned}\quad (46)$$

$$\begin{aligned}(-g)t^{0j} &= \frac{1}{16\pi}\left(\frac{3}{4}\bar{h}^{00}_{,0}\bar{h}^{00}_{,j} + \bar{h}^{00}_{,m}\bar{h}^{0m}_{,j} - \bar{h}^{00}_{,m}\bar{h}^{0j}_{,m} \right. \\ &\quad \left. + \bar{h}^{0j}_{,k}\bar{h}^{kl}_{,l} - \bar{h}^{00}_{,0}\bar{h}^{ij}_{,j} + \bar{h}^{ij}_{,k}\bar{h}^{ik}_{,0} \right. \\ &\quad \left. - \bar{h}^{0i}_{,k}\bar{h}^{ik}_{,j} + \bar{h}^{0i}_{,k}\bar{h}^{ij}_{,k} - \bar{h}^{0i}_{,i}\bar{h}^{jk}_{,k} \right. \\ &\quad \left. - \frac{1}{4}\bar{h}^{kk}_{,0}\bar{h}^{00}_{,j} - \frac{1}{4}\bar{h}^{00}_{,0}\bar{h}^{kk}_{,j} \right. \\ &\quad \left. - \frac{1}{2}\bar{h}^{ik}_{,0}\bar{h}^{ik}_{,j} + \frac{1}{4}\bar{h}^{ii}_{,0}\bar{h}^{kk}_{,j}\right).\end{aligned}\quad (47)$$

Note that the first term of Eq. (46) is the same as Eq. (31), and the first three terms of Eq. (47) are the same as Eq. (32). The additional terms all involve  $\bar{h}^{jk}$ , which was effectively zero in the deDonder gauge because of our slow-motion assumption.

Substituting Eqs. (47) and (42)–(44) into Eq. (30) and integrating gives the rate of change of mass-energy inside the sphere  $\mathcal{V}$  as

$$-\frac{d\mathcal{M}_{\mathcal{V}}}{dt} = \frac{d}{dt} \left[ \left( \frac{7\alpha\gamma}{5} - \frac{9\gamma}{5} + \frac{7\alpha\beta}{5} - \frac{12\beta}{5} - \frac{\alpha}{5} + \frac{1}{10} \right) \times \mathcal{E}_{ij}\mathcal{I}_{ij} \right] + \frac{1}{2} \mathcal{E}_{ij} \frac{d\mathcal{I}_{ij}}{dt}. \quad (48)$$

Again, we have kept only the (external field)  $\times$  (body field) crossterms. As expected, this expression reduces to the deDonder result (35) when  $\alpha = \beta = \gamma = 0$ . Using the same argument for Eq. (48) as for Eq. (35), we can conclude that the first term of Eq. (48) is the rate of change of the interaction energy between the tidal field and the body and that the second term represents the rate of work done by the external field on the body (the “tidal heating”).

Notice the gauge dependence (dependence on  $\alpha, \beta, \gamma$ ) of the rate of change of interaction energy

$$\frac{dE_{\text{int}}}{dt} = \frac{d}{dt} \left[ \left( \frac{7\alpha\gamma}{5} - \frac{9\gamma}{5} + \frac{7\alpha\beta}{5} - \frac{12\beta}{5} - \frac{\alpha}{5} + \frac{1}{10} \right) \times \mathcal{E}_{ij}\mathcal{I}_{ij} \right]. \quad (49)$$

By contrast, the “tidal heating” work done on the body has the same, gauge-invariant value as in Newtonian theory

$$\frac{dW}{dt} = -\frac{1}{2} \mathcal{E}_{ij} \frac{d\mathcal{I}_{ij}}{dt}. \quad (50)$$

#### IV. CONCLUSIONS

In this paper, we have shown that the rate of work done by an external tidal field on a body is independent of how gravitational energy is localized in Newtonian theory and that it is gauge invariant in general relativity. Furthermore, this quantity—which we are calling the “tidal heating”—is given unambiguously by Eq. (4). That the tidal heating should be a well-defined and precise quantity is reasonable, given that its physical effects have been observed in the form of volcanic activity of Jupiter’s moon Io [2,11,12].

There remains one aspect of the uniqueness of the tidal heating that we have not explored: a conceivable (but highly unlikely) dependence of  $dW/dt$  on the choice of energy-momentum pseudotensor in general relativity. The arbitrariness of the pseudotensor is general relativity’s analog of Newtonian theory’s arbitrariness of localization of gravitational energy. The fact that  $dW/dt$  is

independent of the Newtonian energy localization suggests that it may also be independent of the general relativistic pseudotensor. In addition, the clear physical nature of  $dW/dt$  gives further confidence that it must be independent of the pseudotensor. Nevertheless, it would be worthwhile to verify explicitly that  $dW/dt$  is pseudotensor-independent.

#### ACKNOWLEDGMENTS

I thank Kip Thorne for proposing this research problem and for helpful advice about its solution and about the prose of this paper. This research was supported in part by NSF Grant AST-9731698 and NASA Grant NAG5-6840.

- 
- [1] K. S. Thorne and J. B. Hartle, *Phys. Rev. D*, **31**, 1815 (1985).
  - [2] S. J. Peale, P. Cassen, and R. T. Reynolds, *Science* **203**, 894 (1979).
  - [3] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman, San Francisco, 1973).
  - [4] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299 (1980).
  - [5] X.-H. Zhang, *Phys. Rev. D*, **34**, 991 (1986).
  - [6] X.-H. Zhang, *Phys. Rev. D*, **31**, 3130 (1985).
  - [7] K. S. Thorne, *Phys. Rev. D*, **58**, 124031 (1998).
  - [8] E. E. Flanagan, *Phys. Rev. D*, **58**, 124030 (1998).
  - [9] J. R. Wilson and G. J. Mathews, *Phys. Rev. Lett.* **75**, 4161 (1995); J. R. Wilson, G. J. Mathews and P. Maronetti, *Phys. Rev. D* **54**, 1317 (1996); G. J. Mathews, P. Maronetti, and J. R. Wilson, in press (gr-qc/9710140). For the explicit identification of an error in the Wilson-Mathews-Maronetti analysis, which presumably produced the stellar compression, see E. E. Flanagan, *Phys. Rev. D*, submitted (gr-qc/9811132).
  - [10] B. A. Smith *et al.*, *Science* **204**, 951 (1979); L. A. Morabito *et al.*, *Science* **204**, 972 (1979).
  - [11] M. N. Ross and G. Schubert, *Icarus* **64**, 391 (1985).
  - [12] G. W. Ojakangas and D. J. Stevenson, *Icarus* **66**, 341 (1986).
  - [13] J. P. Cox and R. T. Giuli, *Principles of Stellar Structure* v.1, (Gordon and Breach, Science Publishers, New York, 1968).
  - [14] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* 2nd ed., (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).
  - [15] S. Chandrasekhar, *Astrophys. J.* **158**, 45 (1969).



FIG. 1. An example of an isolated, slow-motion body: a star or black hole in a binary system with  $R/a \ll 1$ , where  $R$  is the radius of the body and  $a$  is the separation of the body and companion. The dashed circles indicate the boundaries of the body's vacuum local asymptotic rest frame, the region in which  $M/r \ll 1$  and  $r/\mathcal{L} \ll 1$ . Here,  $r$  is the radial coordinate,  $M$  is the mass of the body, and  $\mathcal{L}$  is the scale of homogeneity of the gravitational field. The boundary of the sphere over which we integrate, denoted by  $\partial\mathcal{V}$ , lies within this region.